Outline

Week 6: Matrix Multiplication and Linear Transformation

Course Notes: 4.1,4.2

Goals: Learn the mechanics of matrix multiplication and linear transformation, and use matrix multiplication to describe linear transformations.

Matrix Anatomy

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

A matrix with 3 rows and 4 columns is a 3 by 4 matrix.

Matrix Anatomy

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

A matrix with 3 rows and 4 columns is a 3 by 4 matrix.

We often write $A = [a_{i,j}]$, where $a_{i,j}$ refers to the particular entry of A in row i, column j.

Matrix Anatomy

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

A matrix with 3 rows and 4 columns is a 3 by 4 matrix.

We often write $A = [a_{i,j}]$, where $a_{i,j}$ refers to the particular entry of A in row i, column j.

Here, $a_{3,2} = 6$

Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix}$$

Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+2 & 2+1 & 3+5 & 4-1 \\ 2+8 & 4+6 & 6+6 & 8+2 \\ 3+3 & 6-1 & 9+2 & 12-3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 & 3 \\ 10 & 10 & 12 & 10 \\ 6 & 5 & 11 & 9 \end{bmatrix}$$

Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+2 & 2+1 & 3+5 & 4-1 \\ 2+8 & 4+6 & 6+6 & 8+2 \\ 3+3 & 6-1 & 9+2 & 12-3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 & 3 \\ 10 & 10 & 12 & 10 \\ 6 & 5 & 11 & 9 \end{bmatrix}$$

Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+2 & 2+1 & 3+5 & 4-1 \\ 2+8 & 4+6 & 6+6 & 8+2 \\ 3+3 & 6-1 & 9+2 & 12-3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 & 3 \\ 10 & 10 & 12 & 10 \\ 6 & 5 & 11 & 9 \end{bmatrix}$$

$$10A = \begin{bmatrix} 10 & 20 & 30 & 40 \\ 20 & 40 & 60 & 80 \\ 30 & 60 & 90 & 120 \end{bmatrix}$$

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans$\times$services and people$\times$services

Output: plans$\times$people

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans$\times$services and people$\times$services

Output: plans$\times$people

	Person 1	Person 2
Plan 1	cost	
Plan 2		
Plan 3		

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	cost	
Plan 2		
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	cost	
Plan 2		
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1		
Plan 2		
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	Row 1 · Col 1	
Plan 2		
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	Row 1 · Col 1	Row 1 · Col 2
Plan 2		
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	Row 1 · Col 1	Row 1 · Col 2
Plan 2	Row 2 · Col 1	
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	Row 1 · Col 1	Row 1 · Col 2
Plan 2	Row 2 · Col 1	Row 2 · Col 2
Plan 3		

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	Row 1 · Col 1	Row 1 · Col 2
Plan 2	Row 2 · Col 1	Row 2 · Col 2
Plan 3	Row 3 · Col 1	

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services

Output: plans×people

	Person 1	Person 2
Plan 1	Row 1 · Col 1	Row 1 · Col 2
Plan 2	Row 2 · Col 1	Row 2 · Col 2
Plan 3	Row 3 · Col 1	Row 3 · Col 2

	Text	Minute	GB
Plan 1	0.10	0.05	1
Plan 2	0	0.05	3
Plan 3	0.01	0.01	10

	Person 1	Person 2
Texts	100	10
Mins	30	120
GB	1	0.5

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 10 & 22 \end{bmatrix}$$

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

$[1, 2, 3] \cdot [1, 2, 0] = 5$

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 10 & 22 \end{bmatrix}$$

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

$[1, 2, 3] \cdot [1, 2, 0] = 5$
$[1, 2, 3] \cdot [0, 1, 3] = 11$

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 10 & 22 \end{bmatrix}$$

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

$[1, 2, 3] \cdot [1, 2, 0] = 5$
$[1, 2, 3] \cdot [0, 1, 3] = 11$
$[2, 4, 6] \cdot [1, 2, 0] = 10$

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 10 & 22 \end{bmatrix}$$

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

$[1, 2, 3] \cdot [1, 2, 0] = 5$
$[1, 2, 3] \cdot [0, 1, 3] = 11$
$[2, 4, 6] \cdot [1, 2, 0] = 10$
$[2, 4, 6] \cdot [0, 1, 3] = 22$

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 10 & 22 \end{bmatrix}$$

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

$[1, 2, 3] \cdot [1, 2, 0] = 5$
$[1, 2, 3] \cdot [0, 1, 3] = 11$
$[2, 4, 6] \cdot [1, 2, 0] = 10$
$[2, 4, 6] \cdot [0, 1, 3] = 22$

Another Example

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} =$$

Another Example

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 4 & 7 \\ 6 & 5 \end{bmatrix}$$

Another Example

$$\begin{bmatrix} 2 & 5 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} =$$

Another Example

$$\begin{bmatrix} 2 & 5 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 11 & 7 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 4 & 2 \end{bmatrix}$$

Wait but... why

$$\begin{bmatrix} 1x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Wait but... why

$$\begin{bmatrix} 1x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 2 \end{array}\right]$$

Wait but... why

$$\begin{bmatrix} 1x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 2 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$Ax = b$$

Dimensions

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

We can only take the dot product of two vectors that have the same length.

Dimensions

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

We can only take the dot product of two vectors that have the same length.

If A is an m-by-n matrix, and B is an r-by-c matrix, then AB is only defined if $n = r$. If $n = r$, then AB is an m-by-c matrix.

Dimensions

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

We can only take the dot product of two vectors that have the same length.

If A is an m-by-n matrix, and B is an r-by-c matrix, then AB is only defined if $n = r$. If $n = r$, then AB is an m-by-c matrix.

Can you always multiply a matrix by itself?

Properties of Matrix Multiplication

One important property DOESN'T hold.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} =$$

Properties of Matrix Multiplication

One important property DOESN'T hold.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} =$$

Properties of Matrix Multiplication

One important property DOESN'T hold.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

Properties of Matrix Multiplication

One important property DOESN'T hold.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

Matrix multiplication is not commutative. *Order matters.*

Properties of Matrix Multiplication

One important property DOESN'T hold.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

Matrix multiplication is not commutative. *Order matters.*

Suppose the matrix product AB exists. Does the product BA also have to exist?

Properties of Matrix Algebra

The other properties hold as you would like. (Page 128, notes.)

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $s(A + B) = sA + sB$
4. $(s + t)A = sA + tA$
5. $(st)A = s(tA)$
6. $1A = A$
7. $A + \mathbf{0} = A$ (where $\mathbf{0}$ is the matrix of all zeros)
8. $A - A = A + (-1)A = \mathbf{0}$
9. $A(B + C) = AB + AC$
10. $(A + B)C = AC + BC$
11. $A(BC) = (AB)C$
12. $s(AB) = (sA)B = A(sB)$

Examples

Simplify the following expressions.

1) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 8 & 9 & 8 \\ 9 & 8 & 9 \\ 8 & 9 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -8 & -9 & -8 \\ -9 & -8 & -9 \\ -8 & -9 & -8 \end{bmatrix}$$

2) $$\left(\begin{bmatrix} 33 & 44 \\ 55 & 66 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 7 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

3) $$2.8 \begin{bmatrix} 15 & 0 & 38 \\ 9 & 10 & 11 \\ 8 & 7 & 6 \end{bmatrix} + 5.6 \begin{bmatrix} -2.5 & 0 & 1 \\ 0.5 & 0 & -0.5 \\ 1 & 1.5 & 2 \end{bmatrix}$$

More on Dimensions

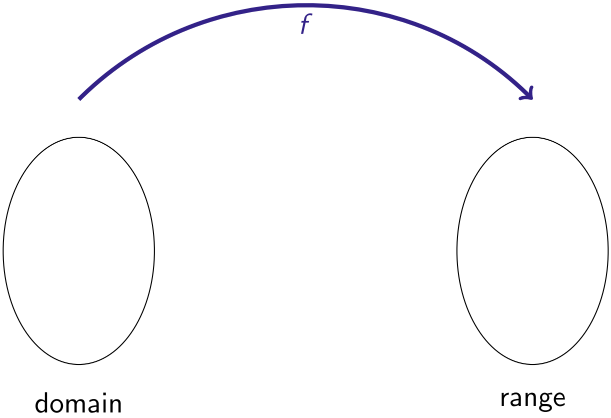
Suppose A is an m-by-n matrix, and B is an r-by-c matrix.

If we want to multiply A and B,
what has to be true about m, n, r, and c?

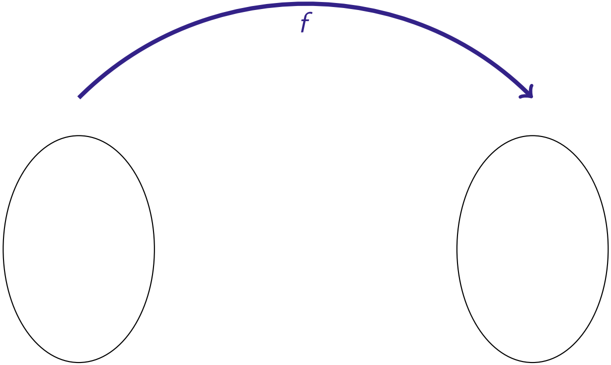
If we want to add A and B,
what has to be true about m, n, r, and c?

If we want to compute $(A + B)A$,
what has to be true about m, n, r, and c?

Functions and Transformations

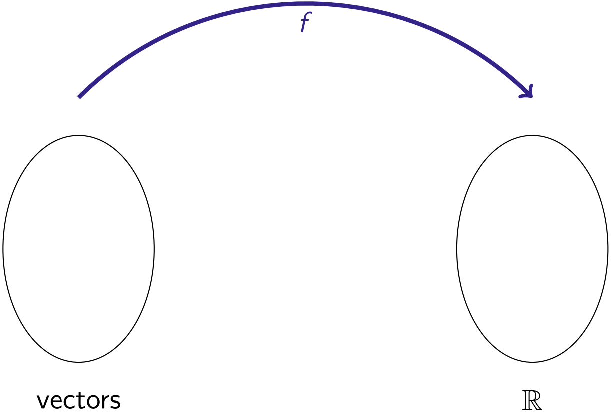


Functions and Transformations



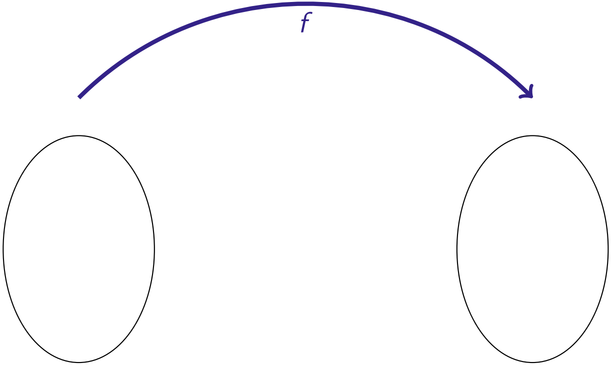
$$f(v) = \|v\|$$

Functions and Transformations



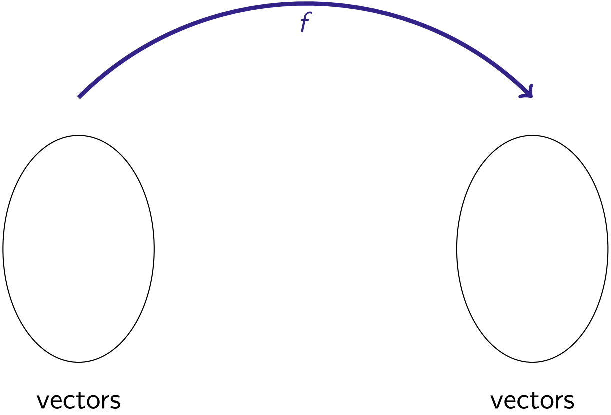
$$f(v) = \|v\|$$

Functions and Transformations



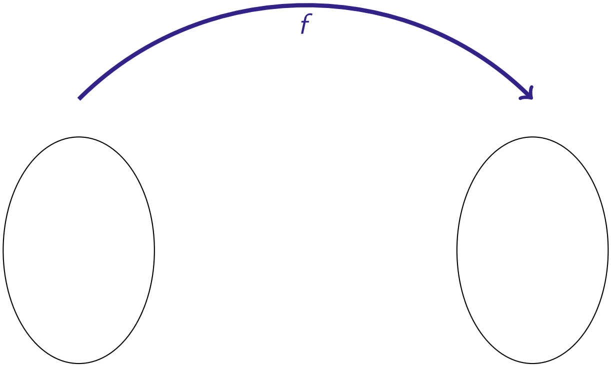
$$f(v) = 3v$$

Functions and Transformations



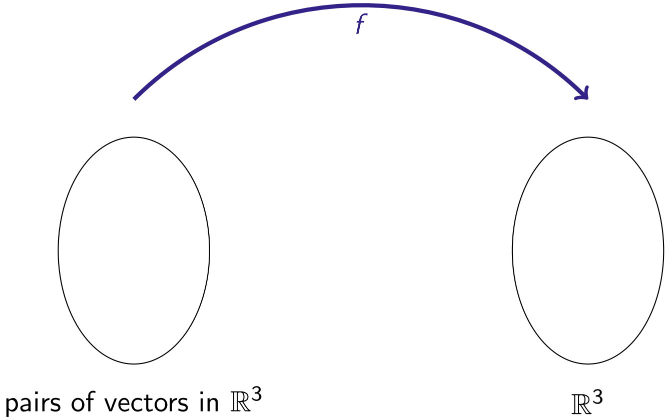
$$f(v) = 3v$$

Functions and Transformations



$$f(u, v) = u \times v$$

Functions and Transformations



$$f(u, v) = u \times v$$

Linear Transformations

$$f(x) = x^2$$

Linear Transformations

$$f(x) = x^2$$

$$f(2+3) = 25$$

$$f(2) + f(3) = 4 + 9 = 13$$

Linear Transformations

$$f(x) = x^2$$

$$f(2+3) = 25 \qquad f(2) + f(3) = 4 + 9 = 13$$

$$f(2 * 3) = 36 \qquad 2f(3) = 2 \cdot 9 = 18$$

Linear Transformations

$$f(x) = x^2$$

$$f(2+3) = 25 \qquad f(2) + f(3) = 4 + 9 = 13$$

$$f(2 * 3) = 36 \qquad 2f(3) = 2 \cdot 9 = 18$$

$$g(x) = 5x$$

$$g(2+3) = 25 \qquad g(2) + g(3) = 10 + 15 = 25$$

Linear Transformations

$$f(x) = x^2$$

$$f(2+3) = 25 \qquad f(2) + f(3) = 4 + 9 = 13$$

$$f(2*3) = 36 \qquad 2f(3) = 2 \cdot 9 = 18$$

$$g(x) = 5x$$

$$g(2+3) = 25 \qquad g(2) + g(3) = 10 + 15 = 25$$

$$g(2*3) = 30 \qquad 2g(3) = 2 \cdot 15 = 30$$

Linear Transformations

$$f(x) = x^2$$

$$f(2+3) = 25 \qquad f(2)+f(3) = 4+9 = 13$$

$$f(2*3) = 36 \qquad 2f(3) = 2\cdot 9 = 18$$

$$g(x) = 5x$$

$$g(2+3) = 25 \qquad g(2)+g(3) = 10+15 = 25$$

$$g(2*3) = 30 \qquad 2g(3) = 2\cdot 15 = 30$$

$$g(x+y) = 5(x+y) = 5x+5y = g(x)+g(y)$$

$$g(xy) = 5(xy) = x(5y) = xg(y)$$

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Is differentiation $T(f(x)) = \frac{d}{dx}[f(x)]$ (of functions whose derivatives exist everywhere) a linear transformation?

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Is differentiation $T(f(x)) = \frac{d}{dx}[f(x)]$ (of functions whose derivatives exist everywhere) a linear transformation?

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 2x \end{bmatrix}$. Is T a linear transformation?

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x \end{bmatrix}$. Is T a linear transformation?

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x \end{bmatrix}$. Is T a linear transformation?

Checking scalar multiplication:

$$T\left(s\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} sx \\ sy \end{bmatrix}\right) = \begin{bmatrix} sx+sy \\ 2sx \end{bmatrix}$$

$$sT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = s\begin{bmatrix} x+y \\ 2x \end{bmatrix} = \begin{bmatrix} sx+sy \\ 2sx \end{bmatrix} = T\left(s\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

So, this property holds

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x \end{bmatrix}$. Is T a linear transformation?

Checking addition:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} a+b+c+d \\ 2(a+c) \end{bmatrix}$$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ 2a \end{bmatrix} + \begin{bmatrix} c+d \\ 2c \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ 2(a+c) \end{bmatrix}$$

So, this property holds

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s, $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(s\mathbf{x}) = sT(\mathbf{x})$.

Are the following linear transformations?

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(\mathbf{x}) = \|\mathbf{x}\|, \ \mathbf{x} \text{ in } \mathbb{R}^2$$

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -1 \\ y \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

There are two ways to do this: checking directly, or using properties of the dot product. First we check directly.

Checking scalar multiplication:

$$S\left(s\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = S\left(\begin{bmatrix} sx \\ sy \\ sz \end{bmatrix}\right) = \begin{bmatrix} sz \\ sy \\ sx \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = sz + 2sy + 3sx =$$

$$s(z + 2y + 3x) = s\left(\begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = sS\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

This property holds.

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Checking addition:

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = S\left(\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}\right) = \begin{bmatrix} c+f \\ b+e \\ a+d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$$

$$(c+f) + 2(b+e) + 3(a+d) = (c+2b+3a) + (f+2e+3d) =$$

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} f \\ e \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + S\left(\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) =$$

$$s\left(\begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = sT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

This property holds.

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

So, S is a linear transformation.

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We can also note that $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Using properties of dot products, for any vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{v}$, and any scalar s:

$$S(s\mathbf{x}) = (s\mathbf{x}) \cdot (v) = s\left(\mathbf{x} \cdot \mathbf{v}\right) = sS(\mathbf{x})$$

$$S(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y}) \cdot (v) = (\mathbf{x} \cdot \mathbf{v}) + (\mathbf{y} \cdot \mathbf{v}) = S(\mathbf{x}) + S(\mathbf{v})$$

So, S is a linear transformation.

$T(\mathbf{x}) = \|\mathbf{x}\|$

Let's remember some logic: a statement is true if it is ALWAYS true, and false if it is EVER false. So, to prove that something IS a linear transformation, we have to show the two properties ALWAYS hold. To show something IS NOT a linear transformation, it is enough to show that ONE of the two properties fails at ONE time.

If $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, then $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{0}) = 0$, but $T(\mathbf{x}) + T(\mathbf{y}) = 1 + 1 \neq 0$. So, T is not a linear transformation.

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -1 \\ y \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Let's remember some logic: a statement is true if it is ALWAYS true, and false if it is EVER false. So, to prove that something IS a linear transformation, we have to show the two properties ALWAYS hold. To show something IS NOT a linear transformation, it is enough to show that ONE of the two properties fails at ONE time.

$R\left(0\begin{bmatrix} x \\ y \end{bmatrix}\right) = R\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (because those vectors aren't parallel)

On the other hand, $0R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 0\left(\begin{bmatrix} x \\ -1 \\ y \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So, R is not a linear transformation.

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Is the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ linear?

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Is the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ linear?

If A is a matrix, then the transformation

$$T(\mathbf{x}) = A\mathbf{x}$$

of a vector $\mathbf{x}$ is linear.

Linear Transformations

Geometric Interpretation

We interpret a matrix geometrically as a **function** from some vectors to some other vectors.

Linear Transformations

Geometric Interpretation

We interpret a matrix geometrically as a **function** from some vectors to some other vectors.

In particular, the function is a **linear transformation**, so it preserves addition and scalar multiplication.

Linear Transformations

Geometric Interpretation

We interpret a matrix geometrically as a **function** from some vectors to some other vectors.
In particular, the function is a **linear transformation**, so it preserves addition and scalar multiplication.

If $T(\mathbf{x}) = A\mathbf{x}$ for some 3×5 matrix A (and a vector $\mathbf{x}$), what are the domain and range of the function T?

Linear Transformations

Geometric Interpretation

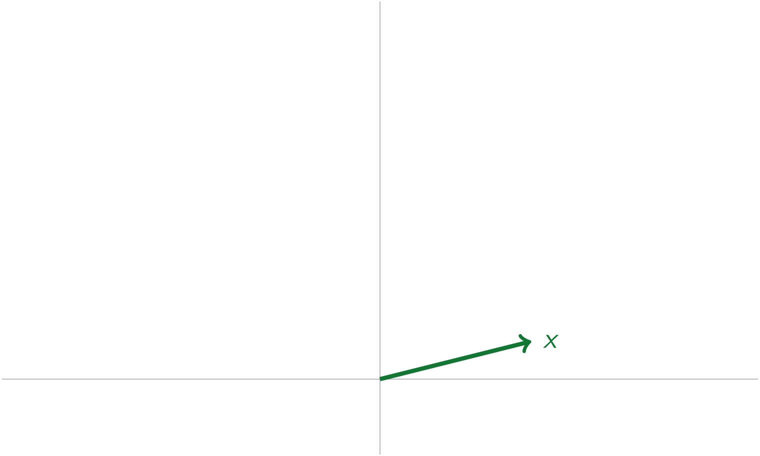
We interpret a matrix geometrically as a **function** from some vectors to some other vectors.
In particular, the function is a **linear transformation**, so it preserves addition and scalar multiplication.

If $T(\mathbf{x}) = A\mathbf{x}$ for some 3×5 matrix A (and a vector $\mathbf{x}$), what are the domain and range of the function T?

If $A\mathbf{x}$ is defined for a vector $\mathbf{x}$, then if A has dimensions $m \times n$, $\mathbf{x}$ must be in $\mathbb{R}^n$, and $A\mathbf{x}$ is in $\mathbb{R}^m$. So, our domain is $\mathbb{R}^n$ and our range is in $\mathbb{R}^m$.

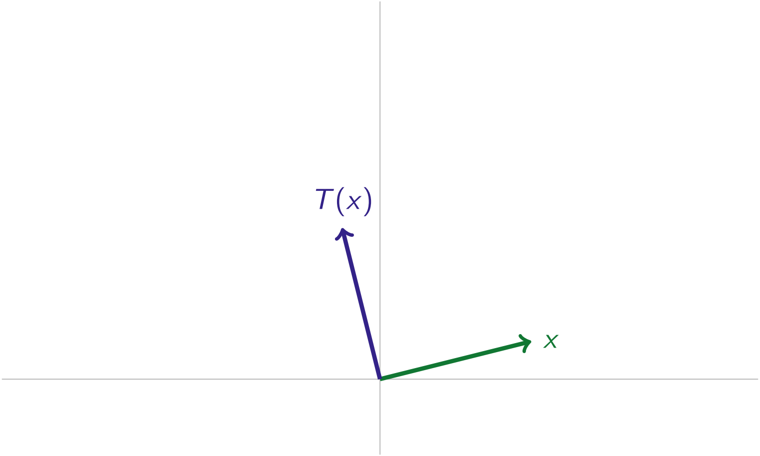
Example

Let $T(x)$ be the rotation of x by ninety degrees.



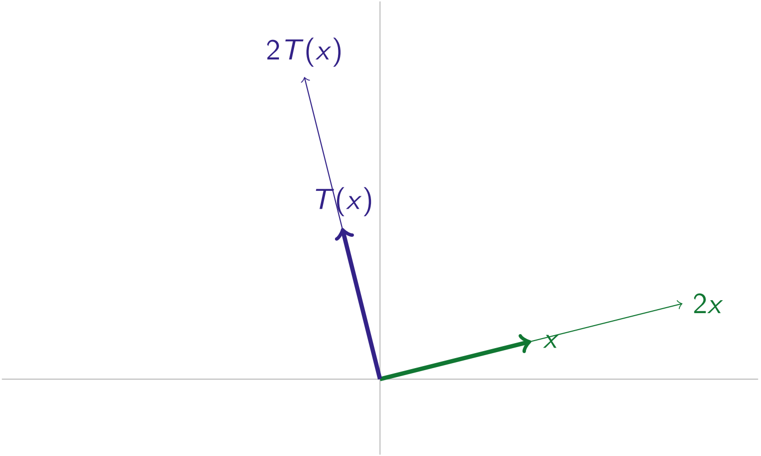
Example

Let $T(x)$ be the rotation of x by ninety degrees.



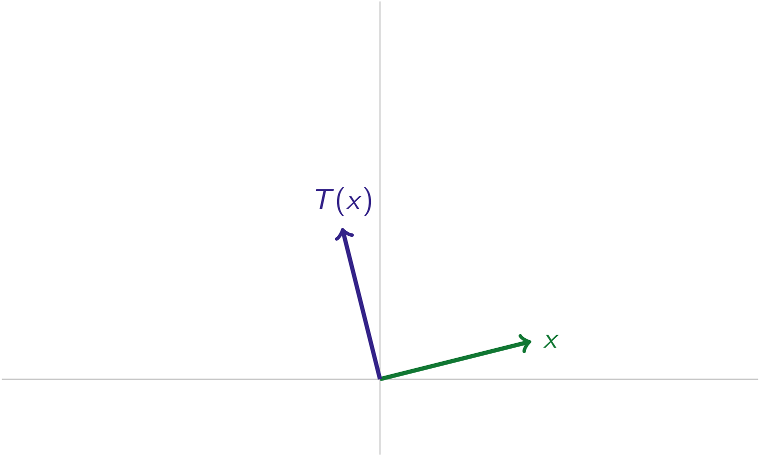
Example

Let $T(x)$ be the rotation of x by ninety degrees.



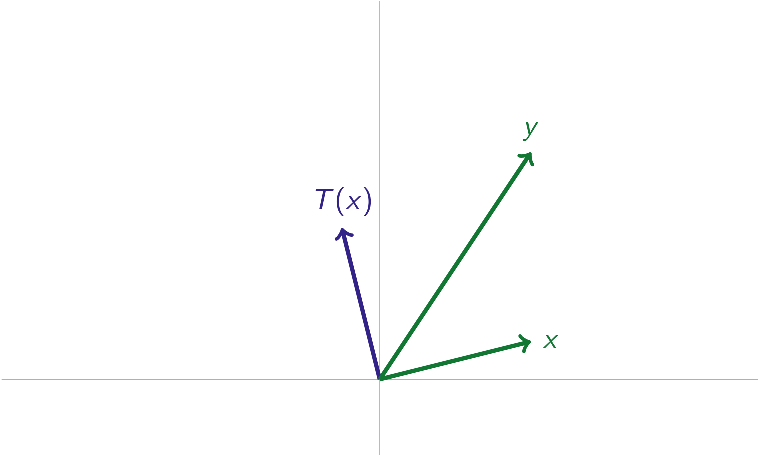
Example

Let $T(x)$ be the rotation of x by ninety degrees.



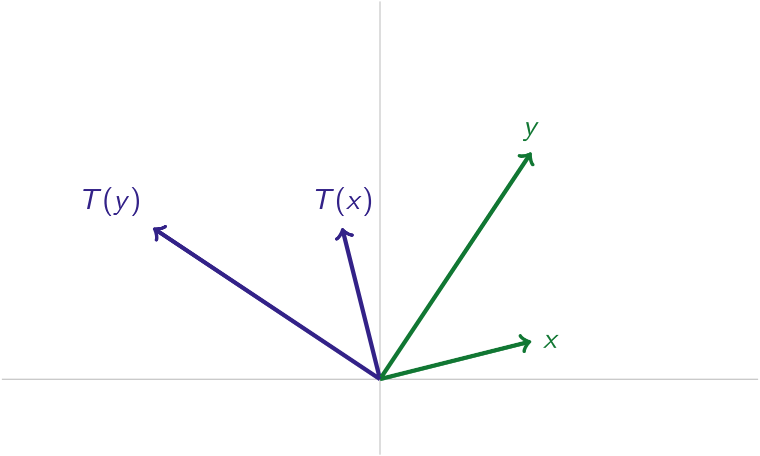
Example

Let $T(x)$ be the rotation of x by ninety degrees.



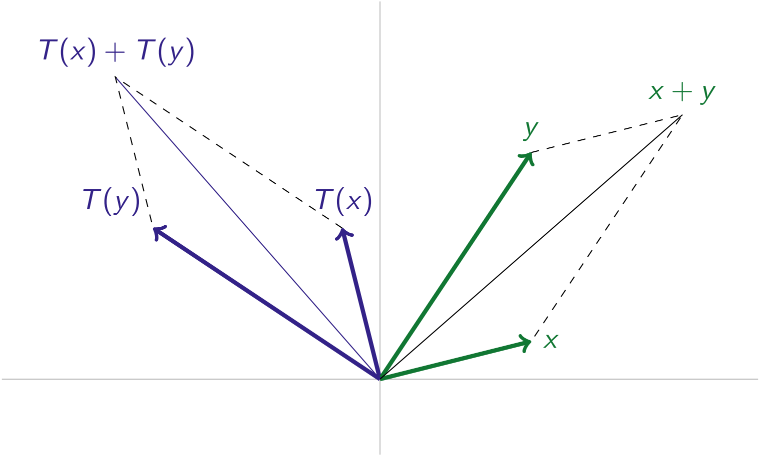
Example

Let $T(x)$ be the rotation of x by ninety degrees.



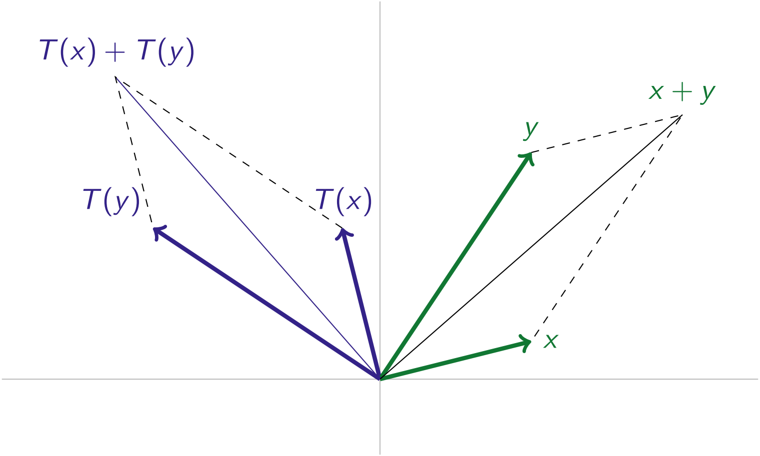
Example

Let $T(x)$ be the rotation of x by ninety degrees.



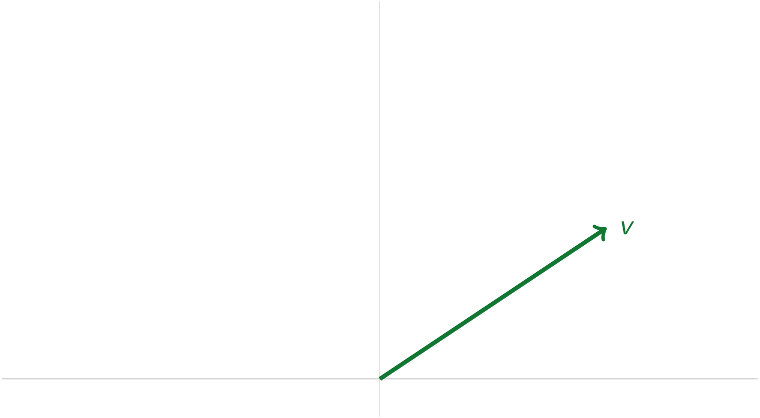
Example

Let $T(x)$ be the rotation of x by ninety degrees.

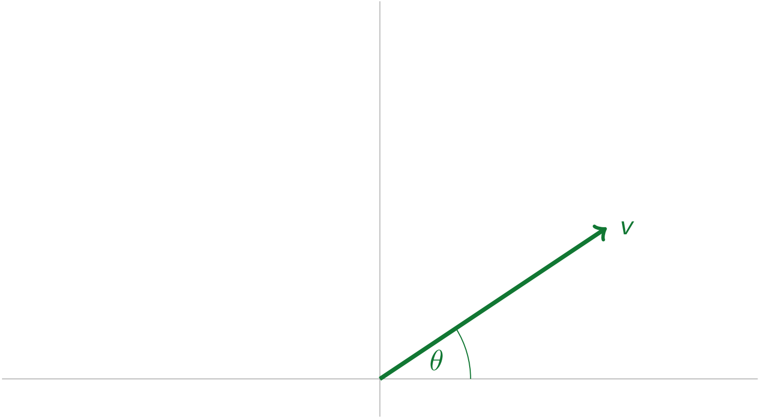


Rotation by a fixed angle is a linear transformation.

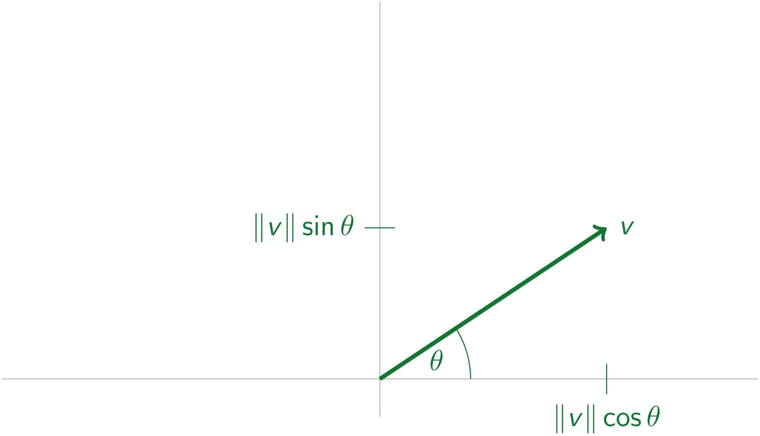
Computing a rotations of ϕ radians (ϕ fixed)



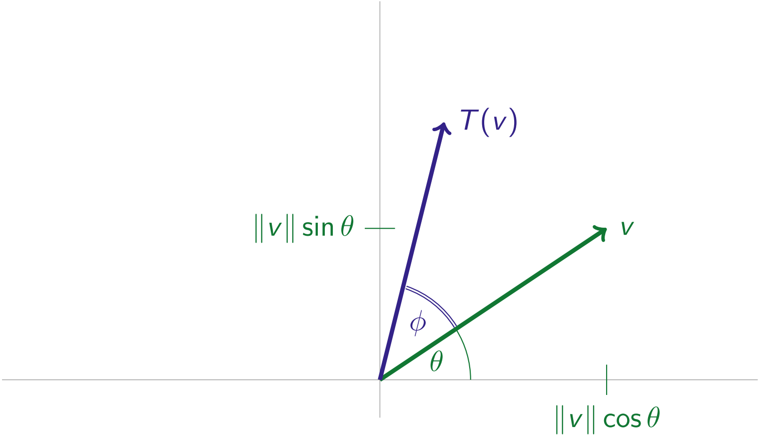
Computing a rotations of ϕ radians (ϕ fixed)



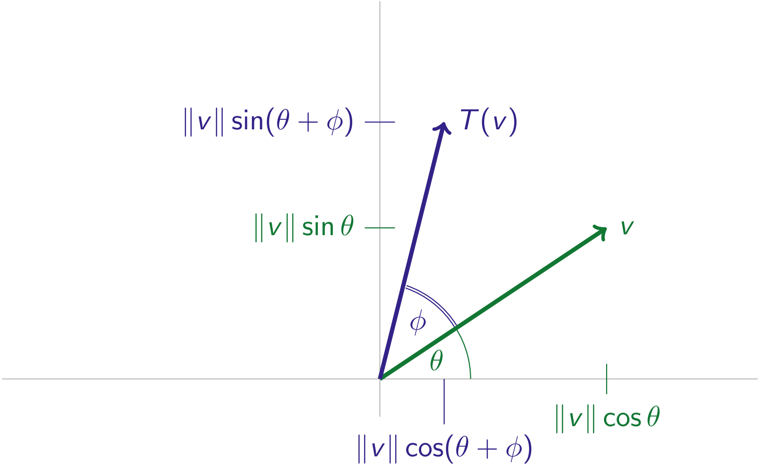
Computing a rotations of ϕ radians (ϕ fixed)



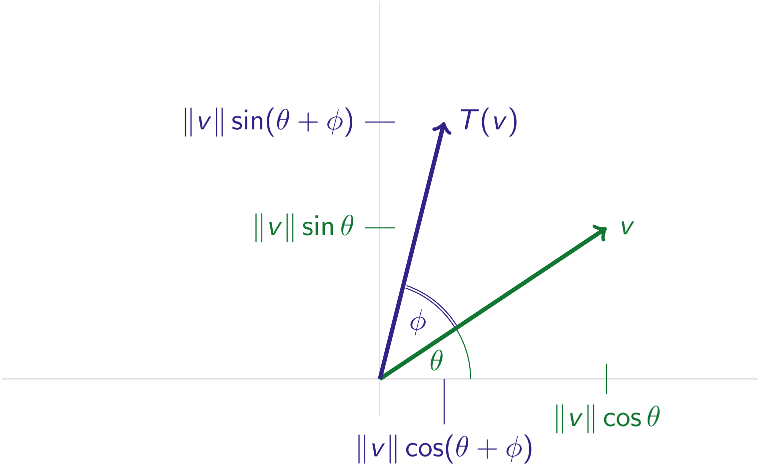
Computing a rotations of ϕ radians (ϕ fixed)



Computing a rotations of ϕ radians (ϕ fixed)

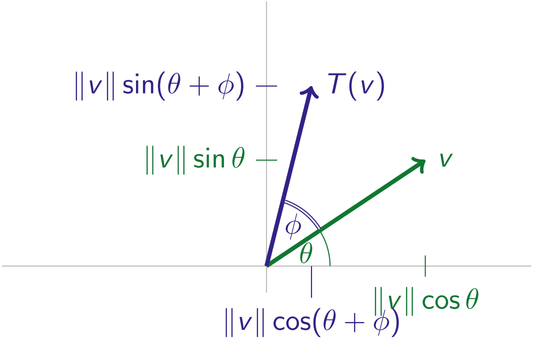


Computing a rotations of ϕ radians (ϕ fixed)



$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ $\qquad$ $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

Computing Rotations



$$v = [v_1, v_2]; \qquad T(v) = [x, y]$$

$$\begin{aligned} x &= \|v\| \cos(\theta + \phi) \\ &= \|v\| (\cos\theta \cos\phi - \sin\phi \sin\theta) \\ &= v_1 \cos\phi - v_2 \sin\phi \end{aligned}$$

$$\begin{aligned} y &= \|v\| \sin(\theta + \phi) \\ &= \|v\| (\sin\theta \cos\phi + \cos\theta \sin\phi) \\ &= v_1 \sin\phi + v_2 \cos\phi \end{aligned}$$

Computing Rotations

$$v = [v_1, v_2]; \qquad T(v) = [x, y]$$

$$\begin{aligned} x &= \|v\| \cos(\theta + \phi) \\ &= \|v\|(\cos\theta\cos\phi - \sin\phi\sin\theta) \\ &= v_1\cos\phi - v_2\sin\phi \end{aligned}$$

$$\begin{aligned} y &= \|v\| \sin(\theta + \phi) \\ &= \|v\|(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= v_1\sin\phi + v_2\cos\phi \end{aligned}$$

Computing Rotations

$$v = [v_1, v_2]; \qquad T(v) = [x, y]$$

$$\begin{aligned} x &= \|v\| \cos(\theta + \phi) \\ &= \|v\|(\cos\theta\cos\phi - \sin\phi\sin\theta) \\ &= v_1\cos\phi - v_2\sin\phi \end{aligned} \qquad \begin{aligned} y &= \|v\| \sin(\theta + \phi) \\ &= \|v\|(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= v_1\sin\phi + v_2\cos\phi \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Computing Rotations

$$v = [v_1, v_2]; \qquad T(v) = [x, y]$$

$$\begin{aligned} x &= \|v\| \cos(\theta + \phi) \\ &= \|v\| (\cos\theta \cos\phi - \sin\phi \sin\theta) \\ &= v_1 \cos\phi - v_2 \sin\phi \end{aligned}$$

$$\begin{aligned} y &= \|v\| \sin(\theta + \phi) \\ &= \|v\| (\sin\theta \cos\phi + \cos\theta \sin\phi) \\ &= v_1 \sin\phi + v_2 \cos\phi \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The matrix is called a rotation matrix, Rot_ϕ

Computing Rotations

$$v = [v_1, v_2]; \qquad T(v) = [x, y]$$

$$\begin{aligned} x &= \|v\| \cos(\theta + \phi) \\ &= \|v\|(\cos\theta \cos\phi - \sin\phi \sin\theta) \\ &= v_1 \cos\phi - v_2 \sin\phi \end{aligned} \qquad \begin{aligned} y &= \|v\| \sin(\theta + \phi) \\ &= \|v\|(\sin\theta \cos\phi + \cos\theta \sin\phi) \\ &= v_1 \sin\phi + v_2 \cos\phi \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The matrix is called a rotation matrix, Rot_ϕ

Computationally nice! Compute the constants in the matrix only one time, then you can rotate any vector you like, in the entire xy-plane.

Computing Rotations

$$\text{Rot}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 90 degrees ($\pi/2$ radians)?

Computing Rotations

$$\text{Rot}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 90 degrees ($\pi/2$ radians)?

$$\text{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Computing Rotations

$$\mathrm{Rot}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 90 degrees ($\pi/2$ radians)?

$$\mathrm{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 30 degrees ($\pi/6$ radians)?

Computing Rotations

$$\mathrm{Rot}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 90 degrees ($\pi/2$ radians)?

$$\mathrm{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 30 degrees ($\pi/6$ radians)?

$$\mathrm{Rot}_{\pi/6} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

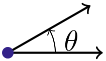
Are rotations commutative?

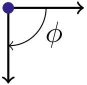
Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?





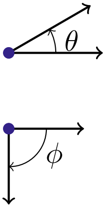
Are rotations commutative?

Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?



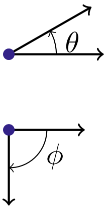
Are rotations commutative?

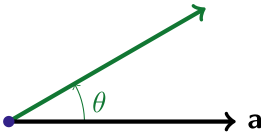
Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?





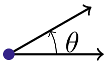
Are rotations commutative?

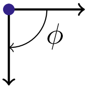
Let **a** be a vector in $\mathbb{R}^2$.

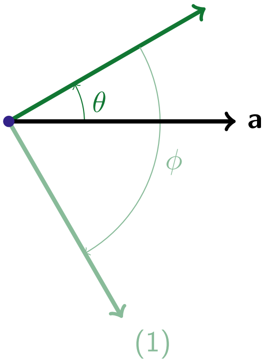
(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?







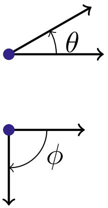
Are rotations commutative?

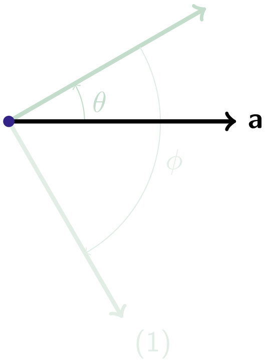
Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?





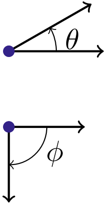
Are rotations commutative?

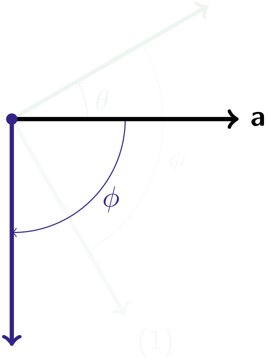
Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?





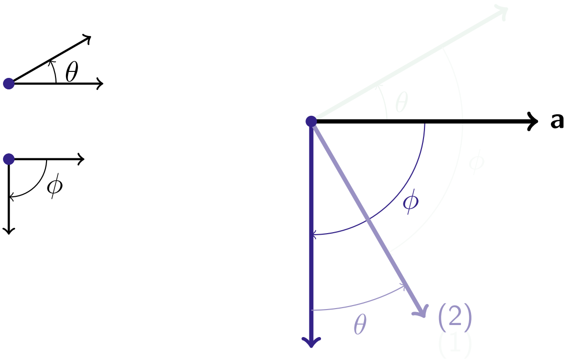
Are rotations commutative?

Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?



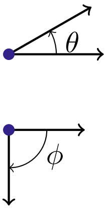
Are rotations commutative?

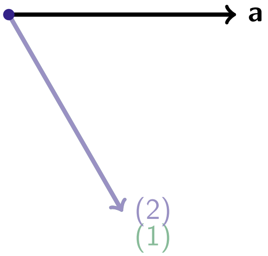
Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?





Are rotations commutative?

Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?

Are rotations commutative?

Let $\mathbf{a}$ be a vector in $\mathbb{R}^2$.

(1) Rotate the vector $\mathbf{a}$ by θ radians, then by ϕ radians.

(1a) $\mathbf{a}$ rotated by θ radians is $\text{Rot}_\theta \mathbf{a}$.

(2) Rotate the vector $\mathbf{a}$ by ϕ radians, then by θ radians.

Will you always end up with the same thing?

Are rotations commutative?

Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(1a) **a** rotated by θ radians is $\text{Rot}_\theta \mathbf{a}$.

(1b) $\text{Rot}_\theta \mathbf{a}$ rotated by ϕ radians is $\text{Rot}_\phi(\text{Rot}_\theta \mathbf{a})$

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?

Are rotations commutative?

Let $\mathbf{a}$ be a vector in $\mathbb{R}^2$.

(1) Rotate the vector $\mathbf{a}$ by θ radians, then by ϕ radians.

(1a) $\mathbf{a}$ rotated by θ radians is $\mathrm{Rot}_\theta \mathbf{a}$.

(1b) $\mathrm{Rot}_\theta \mathbf{a}$ rotated by ϕ radians is $\mathrm{Rot}_\phi (\mathrm{Rot}_\theta \mathbf{a})$

(2) Rotate the vector $\mathbf{a}$ by ϕ radians, then by θ radians.

(2a) $\mathbf{a}$ rotated by ϕ radians is $\mathrm{Rot}_\phi \mathbf{a}$.

Will you always end up with the same thing?

Are rotations commutative?

Let $\mathbf{a}$ be a vector in $\mathbb{R}^2$.

(1) Rotate the vector $\mathbf{a}$ by θ radians, then by ϕ radians.

- (1a) $\mathbf{a}$ rotated by θ radians is $\text{Rot}_\theta \mathbf{a}$.
- (1b) $\text{Rot}_\theta \mathbf{a}$ rotated by ϕ radians is $\text{Rot}_\phi (\text{Rot}_\theta \mathbf{a})$

(2) Rotate the vector $\mathbf{a}$ by ϕ radians, then by θ radians.

- (2a) $\mathbf{a}$ rotated by ϕ radians is $\text{Rot}_\phi \mathbf{a}$.
- (2b) $\text{Rot}_\phi \mathbf{a}$ rotated by θ radians is $\text{Rot}_\theta (\text{Rot}_\phi \mathbf{a})$

Will you always end up with the same thing?

Are rotations commutative?

Let $\mathbf{a}$ be a vector in $\mathbb{R}^2$.

(1) Rotate the vector $\mathbf{a}$ by θ radians, then by ϕ radians.

(1a) $\mathbf{a}$ rotated by θ radians is $\text{Rot}_\theta\mathbf{a}$.

(1b) $\text{Rot}_\theta\mathbf{a}$ rotated by ϕ radians is $\text{Rot}_\phi\left(\text{Rot}_\theta\mathbf{a}\right)$

(2) Rotate the vector $\mathbf{a}$ by ϕ radians, then by θ radians.

(2a) $\mathbf{a}$ rotated by ϕ radians is $\text{Rot}_\phi\mathbf{a}$.

(2b) $\text{Rot}_\phi\mathbf{a}$ rotated by θ radians is $\text{Rot}_\theta\left(\text{Rot}_\phi\mathbf{a}\right)$

Will you always end up with the same thing?

That is, will

$$\text{Rot}_\phi\left(\text{Rot}_\theta\mathbf{a}\right) = \text{Rot}_\theta\left(\text{Rot}_\phi\mathbf{a}\right)$$

for every θ, every ϕ, and every $\mathbf{a}$ in $\mathbb{R}^2$?

Are rotations commutative?

Will

$$\mathrm{Rot}_\phi\left(\mathrm{Rot}_\theta \mathbf{a}\right) = \mathrm{Rot}_\theta\left(\mathrm{Rot}_\phi \mathbf{a}\right)$$

for every θ, every ϕ, and every $\mathbf{a}$ in $\mathbb{R}^2$?

Are rotations commutative?

Will

$$\mathrm{Rot}_{\phi}\left(\mathrm{Rot}_{\theta}\mathbf{a}\right) = \mathrm{Rot}_{\theta}\left(\mathrm{Rot}_{\phi}\mathbf{a}\right)$$

for every θ, every ϕ, and every **a** in $\mathbb{R}^2$?

In general, matrix multiplication is not commutative, but we don't care about ALL matrices–only rotation matrices.

Are rotations commutative?

Will

$$\mathrm{Rot}_{\phi}\left(\mathrm{Rot}_{\theta}\mathbf{a}\right) = \mathrm{Rot}_{\theta}\left(\mathrm{Rot}_{\phi}\mathbf{a}\right)$$

for every θ, every ϕ, and every **a** in $\mathbb{R}^2$?
In general, matrix multiplication is not commutative, but we don't care about ALL matrices–only rotation matrices.

$$\mathrm{Rot}_{\phi}\mathrm{Rot}_{\theta} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathrm{Rot}_{\theta}\mathrm{Rot}_{\phi} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Are rotations commutative?

Will

$$\mathrm{Rot}_\phi\left(\mathrm{Rot}_\theta \mathbf{a}\right) = \mathrm{Rot}_\theta\left(\mathrm{Rot}_\phi \mathbf{a}\right)$$

for every θ, every ϕ, and every **a** in $\mathbb{R}^2$?

In general, matrix multiplication is not commutative, but we don't care about ALL matrices–only rotation matrices.

$$\mathrm{Rot}_\phi \mathrm{Rot}_\theta = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathrm{Rot}_\theta \mathrm{Rot}_\phi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\mathrm{Rot}_{\theta+\phi} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & (\cos\theta+\phi) \end{bmatrix}$$